

Nerves, ∞ -categories and the
Boardman-Vogt construction.

∞ -category theory reading group, ntop server

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Plan

If time
permits

- Segal category

- The Nerve construction

- Def and Examples

- Grothendieck Segal condition

- ∞ -categories

- Def and Examples

- ∞ -groupoids

- Op -category construction

- Boardman-Vogt - universal construction

- The realization functor

- Joyal's coherence lemma and
the Homotopy
cat.-const.

Def: The nerve associated to a category is the s.set.

$$NC_n := \text{Hom}_{\text{Cat}}([n], C)$$

→ category of small categories

We can think as follows

$$NC_n := \{ \text{strings of } n\text{-composable arrows in } C \}$$

The degeneracy map $s_i: NC_n \rightarrow NC_{n+1}$
takes a string of n -composable arrows

to a $n+1$ -string by attaching an id-mp at the i^{th} place.

$$C_0 \xrightarrow{b_1} C_1 \xrightarrow{b_2} \dots \xrightarrow{b_i} C_i \xrightarrow{b_{i+1}} \dots \xrightarrow{b_n} C_n$$

$$\downarrow s_i$$

$$C_0 \xrightarrow{b_1} C_1 \xrightarrow{b_2} \dots \xrightarrow{b_i} C_i \xrightarrow{\text{Id}} C_i \xrightarrow{b_{i+1}} C_{i+1} \xrightarrow{\dots} C_n$$

and the face map $d_i: NC_n \rightarrow NC_{n-1}$ compares the i^{th} and the $(i+1)^{\text{th}}$ arrows for $\delta \in \Delta_n$

and if $i=0, n$, delete i^{th} object.

$$C_0 \xrightarrow{b_1} C_1 \xrightarrow{b_2} C_2 \rightarrow \dots \rightarrow C_{i-1} \xrightarrow{b_i} C_i \xrightarrow{b_{i+1}} C_{i+1} \rightarrow \dots \xrightarrow{b_n} C_n$$

↓
diagonal

$$C_0 \xrightarrow{b_1} C_1 \xrightarrow{b_2} C_2 \rightarrow \dots \rightarrow C_{i-1} \xrightarrow{b_{i+1} \circ b_i} C_{i+1} \xrightarrow{b_{i+2}} \dots \xrightarrow{b_n} C_n$$

We've a fully faithful functor

$$i: \Delta \rightarrow \text{Set}$$

The nerve functor is just evaluation at i

$$N = i^*: \text{Cat} \rightarrow \text{Set}$$

By Kan's theorem, it has a left adjoint:

$$\tau: \text{Set} \rightarrow \text{Cat}$$

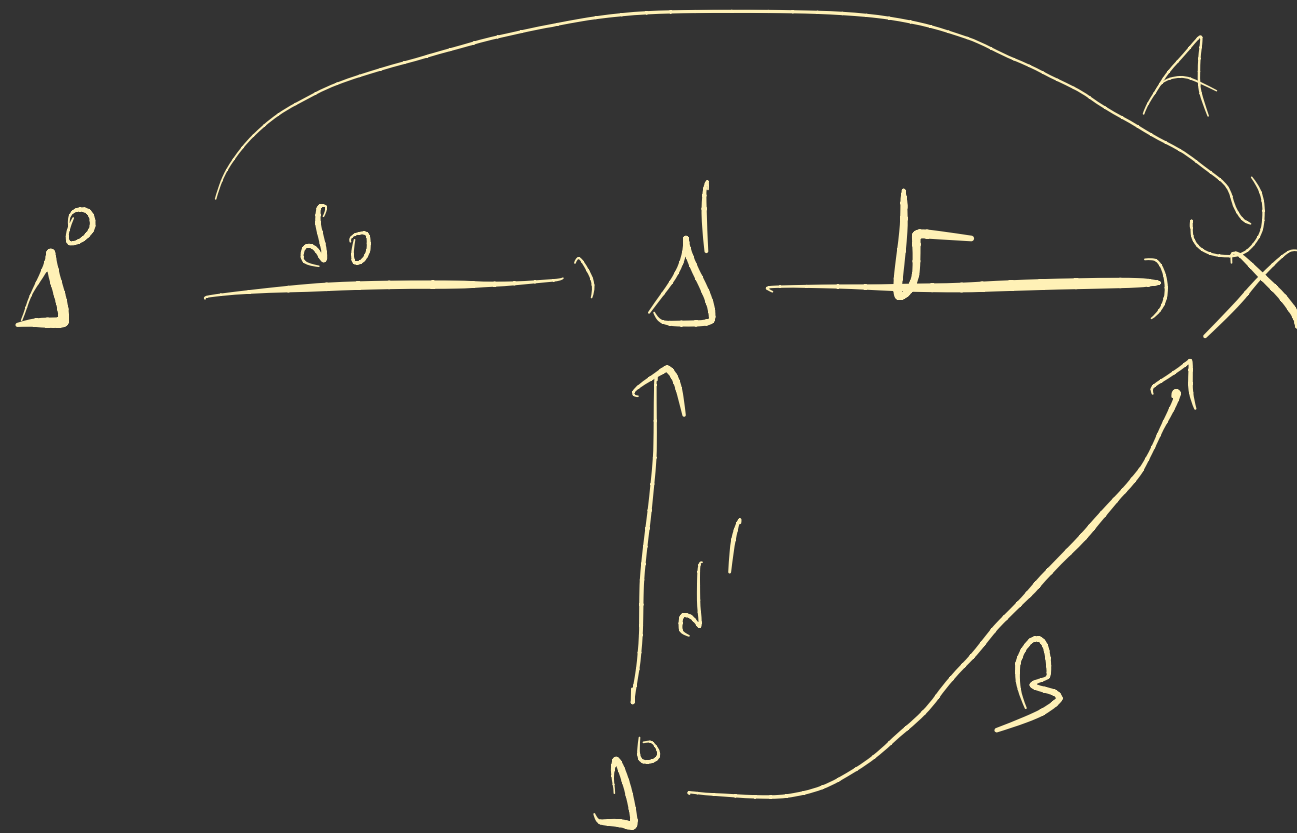
$$(\tau, N): \text{Set} \rightleftarrows \text{Cat}$$

Def: For a s-set X , the objects are 0-simplices
(i.e. $\Delta^0 \rightarrow X$) and morphisms are

1-simplices (i.e. $\Delta^1 \rightarrow X$)

$$A \xrightarrow{k} B \quad (\text{an edge in } X)$$

the faces of k are given by
target $\text{to } k = B$ & source $\text{of } k = A$



For an object x of X , the dependent edge
 $s_0(x) : \ast$ is the morphism $X \rightarrow \ast$

Recall: For $n \in \mathbb{Z}^{\geq 0}$ Δ^n is the s -set which is

given by the following constant function

$$([\mu] \in \Delta) \mapsto (\text{Item}_\Delta([\mu], [\mu]))$$

Remark For $n \in \mathbb{Z}^{\geq 0}$ $N(n)$ can be identified with

$$\Delta^n.$$

For a fin. totally ordered set E we denote

$$\Delta^E = N(E)$$

Now
$$\partial \Delta^n = \bigcup_{E \subsetneq [n]} \Delta^E \subset \Delta^n$$

$$\bigwedge_{k \in E \neq N} \Delta^k = \bigcup_{\substack{\Delta^E \subset \Delta^{\wedge} \\ \sigma \leq k \leq \sigma + 1}} \Delta^{\wedge}$$

Def cell: Spine is a s. set of Δ^{\wedge} where k are monotone maps
 - simplicial
 $\sigma: [k] \rightarrow [n]$ with the $\sigma(k) \leq \sigma(0) + 1$

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \dots \longrightarrow n$$

$$\text{Spine } \tau(n) = \bigcup_{0 \leq j \leq n} \Delta^{\{[i, j], [j, n]\}} \subset \Delta^{\wedge}$$

Def: A simp. obj. X in a category \mathcal{C} is a s.set internl to \mathcal{C} .

E.g (Čech nerve)

let's consider a category \mathcal{C} with pullbacks

and $V \rightarrow Z \in \text{Mor}(\mathcal{C})$

The Čech nerve is a simp. obj. in \mathcal{C}

$$C[1] := \left(\begin{array}{c} \begin{array}{c} V \\ \downarrow \\ V \times_Z V \end{array} \\ \downarrow \\ V \times_Z V \times_Z V \\ \downarrow \\ \vdots \end{array} \right)$$

Def. A triangle in a set X is a map

$$f: \partial\Delta^2 \rightarrow X$$

It can be visualized (high) with

$f, g, h \in \text{Map}(X)$, with

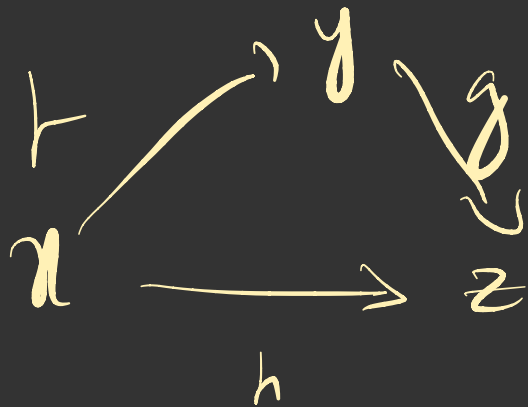
target of f coinciding with the source of g ,
source of f and h are the same,
 g and h have the same target

$$\partial\Delta^2 := \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,2\}}$$

f correspond to the map

$$\Delta' \cong \Delta^{2,0,1,1} \subset \partial \Delta^2 \rightarrow X$$

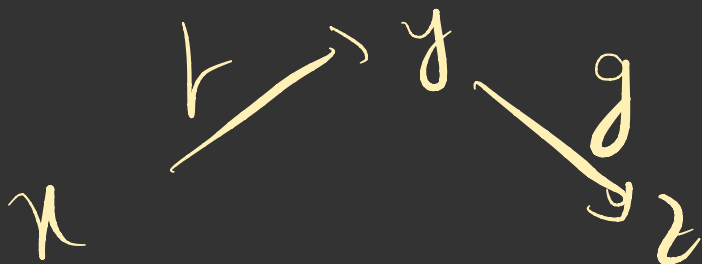
$$\Delta'' \cong \Delta^{2,1,2,1} \subset \partial \Delta^2 \rightarrow X$$



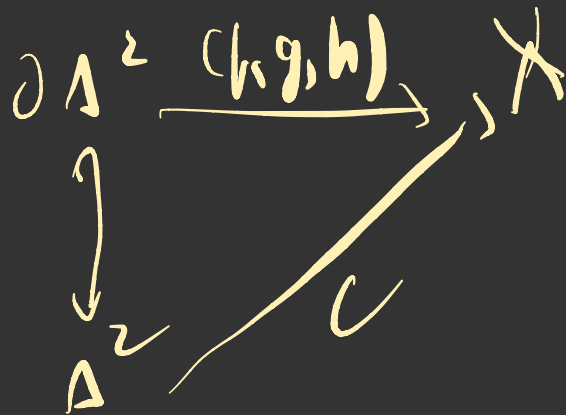
h



$$\partial \wedge_{\mathbb{H}}^2 \mathbb{R}^2 = \text{Spin}(2) \rightarrow X$$



Def: A triangle (f, g, h) is said to commute if
 \exists a morphism $c: \Delta^2 \rightarrow X$ where restricted to
 boundary coincides with (f, g, h)



Courteney - Seifert condition

A S-set X satisfies the Courteney k -
 Seifert condition if the restriction along
 the inclusion $\text{Spine}(n) \subset \Delta^n$ induces a

bijection

$$\text{Hom}(\Delta^n, X) \xrightarrow{\sim} \text{Hom}(S^1, X)$$

$\forall n \in \mathbb{Z}$

consequence If $X \cong \mathbb{N}\mathbb{C}$

$$\text{Hom}(\Delta^n, \mathbb{N}\mathbb{C}) \xrightarrow{\sim} \text{Hom}(S^1, \mathbb{N}\mathbb{C})$$

Proof. The above functor is fully faithful.

Proof.

$$h_1 : \text{Hom}_{\text{Set}}(\mathbb{C}, \mathbb{D}) \rightarrow \text{Hom}_{\text{Set}}(\mathbb{N}\mathbb{C}, \mathbb{N}\mathbb{D})$$

A 11. Sketch This also follows from the

fact that the nerve construction
gives a 2-skeletal s-set.

Brief form

For the complex set Δ with a
subset $\Delta_{\leq 1} \subset \Delta_n$

objects are $\{0, \dots, n\}$

Then $\Delta|_{\leq 1} \hookrightarrow \Delta$ induces a functor

functor

$$\text{tr}_n: \text{sSet} \longrightarrow \text{sSet}_{\leq 1} = (\Delta_{\leq 1}^{\text{op}}, \text{Set})$$

It has fully faithful left adj

$$sk_n: \mathcal{S}et_{\leq n} \longrightarrow \mathcal{S}et$$

& fully faithful right adj:

$$co sk_n: \mathcal{S}et_{\leq n} \longrightarrow \mathcal{S}et \quad \text{range of}$$

Def: \mathcal{S} -sets isomorph w.c. to objects in $\mathcal{P}re_n$ $co sk_n$ are n -skeletal

$$\text{Hom}_{\mathcal{S}et} (NC, ND) \cong \text{Hom}_{\mathcal{S}et_{\leq 2}} (NC|_{\Delta_{\leq 2}^{op}}, ND|_{\Delta_{\leq 2}^{op}})$$

Prop -

The follo^y conditions are equivalent:

$X \in \text{Set}$

(i) \exists a small cat C and an iso-

$$X \cong N(C)$$

(ii) $X \rightarrow N(\tau(X))$ is invertible

(iii) X satisfies the bournthen dieck
-syd -ursh.

Defining ∞ -cat

Def: A set is a Kan-complex if every

horn $\Delta_n^u \rightarrow X$ can extend to the
 $(0 \leq k \leq n)$ n -cunph



Fact: The category of chain complexes is Cartesian closed.

(fin. prod.
 $a \in \text{Obj}(\mathcal{C})$
 $\rightarrow \times a: C \rightarrow SC$ has
 adj.)

Def: An ∞ -category is a weak chain-complex

i.e. an extension of $A_k^n \rightarrow X$
 non-occure

to $\Delta^n \rightarrow X$ exist

Equivalently: Inner Λ_k^n (occure) have holes
 horns

Further Equivalently: An ω -cat is a sset with
adj right lifting property for inner
 horns

for inner horns $\Lambda_k^n \hookrightarrow \Delta^n$ occure

Concept: $\text{Fun}(\Delta^n, X) \longrightarrow \text{Fun}(\Lambda_{\hat{u}}, X)$
is surjective if X is
an ∞ -cat

- For an ∞ -category C objects are vertices
 $x \in C_0$, morphisms are 1-simplices in C ,

If $f \in C_1$ then $X = d_1(f)$ as $\mathcal{M} = \text{dot } 1$.
 $x \rightarrow y$

- A. J. Lurie. edge for $x \in C_0$, $\text{iso}(x)$
ident. by morphisms for x

Def: (∞ -groupoid)

An ∞ -groupoid is an ∞ -category \mathcal{C} where
every morphism is invertible

$$\forall f: X \rightarrow Y \in \text{Mor}(\mathcal{C})$$

$$\exists g: Y \rightarrow X, h: Y \rightarrow X \text{ s.t.}$$

$$\begin{array}{ccc} & Y & \\ f \swarrow & & \searrow g \\ X & = & X \end{array}$$

$$\begin{array}{ccc} & X & \\ h \nearrow & & \searrow f \\ Y & = & Y \end{array}$$

Examples of ∞ -cat

(1) For a small cat \mathcal{C} , \mathcal{NC} is an ∞ -cat.

Obj \rightarrow Obj of \mathcal{C}
Morph \rightarrow morph of \mathcal{C}

(2) For $X \in \text{Top}$, $\text{Sing}(X)$ is an ∞ -cat.

Identity (Δ^1) with the hypercube $(0,1)^n$

Obj \rightarrow points of X

Morph \rightarrow cont. paths $[-, (0,1) \rightarrow X$

source $\downarrow (0)$

target $\downarrow (1)$

$\text{Id} \in X \rightarrow X$ taking value at X

Fact we've the following adjunction formula.

$$\text{Hom}(\mathbb{K}, X) \simeq \text{Hom}(\mathbb{K}, \text{Sing}(X))$$

Surjection
along $\Lambda_{\mathbb{K}}^{\wedge} \subseteq \Delta^{\wedge}$

$$\text{Hom}(\Delta^{\wedge}, \text{Sing}(X)) \longrightarrow \text{Hom}(\Lambda_{\mathbb{K}}^{\wedge}, \text{Sing}(X))$$

Prop: Every Kan complex is an ∞ -groupoid

Proof: $\forall A \rightarrow B \in \text{Mor}(X)$ $x \in \text{Kan}$

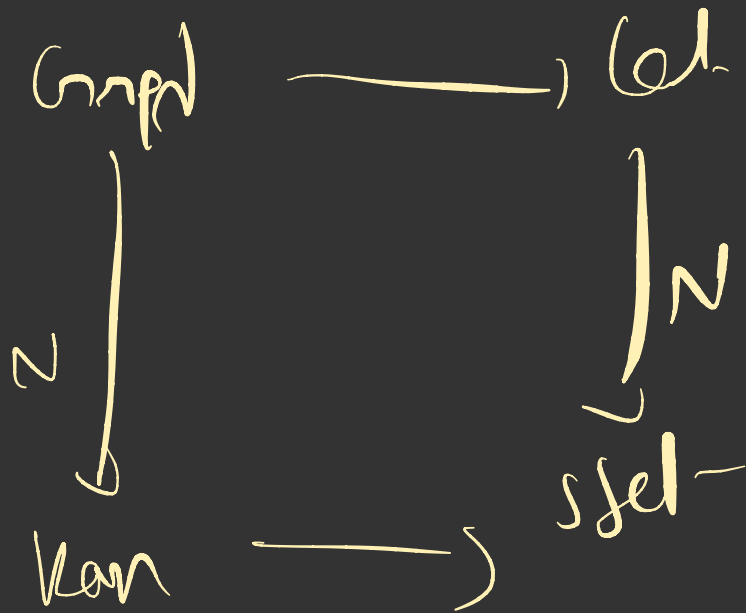
\Rightarrow a unique morphism

$$\Delta^2 \rightarrow A$$

Surds; non-degen. Δ simple $\Delta^{20,13}$ $\Delta^{20,11}$ to Δ



Let Group be the category of groups
then we've the following comm. diagrams



Construkt let's define a set $X := \Delta^{op} \rightarrow \text{Set}$

$$\Delta^{op} \xrightarrow{\text{OP}} \Delta^{op} \xrightarrow{X} \text{Set}$$

For $\alpha: [m] \rightarrow [n]$ in Δ , the morphism

given by $OP(\alpha): [m] \rightarrow [n]$

$$OP(\alpha)(j) = n - \alpha(m-j)$$

Proof: For a small category C ,

$$N(C^{op}) = N(C)^{op}$$

n-simplicial NC on

diagrams

$$c_0 \xrightarrow{h_1} c_1 \rightarrow \dots \xrightarrow{h_n} c_n \text{ in } \mathcal{C}$$

Then diagram gives a construction of

$$c_n \xrightarrow{h_n} c_{n-1} \rightarrow \dots \xrightarrow{h_1} c_0 \text{ in } \mathcal{C}^{\text{op}}$$

$$N(\mathcal{C}^{\text{op}}) = N(\mathcal{C})^{\text{op}} \text{ up to an iso}$$

Boardman-Vogt construction

Def: For any $X \in \text{Top}$ define $\pi_{\leq 1}(X)$ to be the fundamental sfd. of X with

Obj \rightarrow points in X

For $n, m \in \mathbb{N}$ | from $\pi_{\leq 1}(X)$ (n, m can be ident.) with n vertices class of cont. paths

$P^n = P(0, 1) \rightarrow X$ with $P(0) = x$, $P(1) = y$

— composition can be by concatenation of paths

Refinement

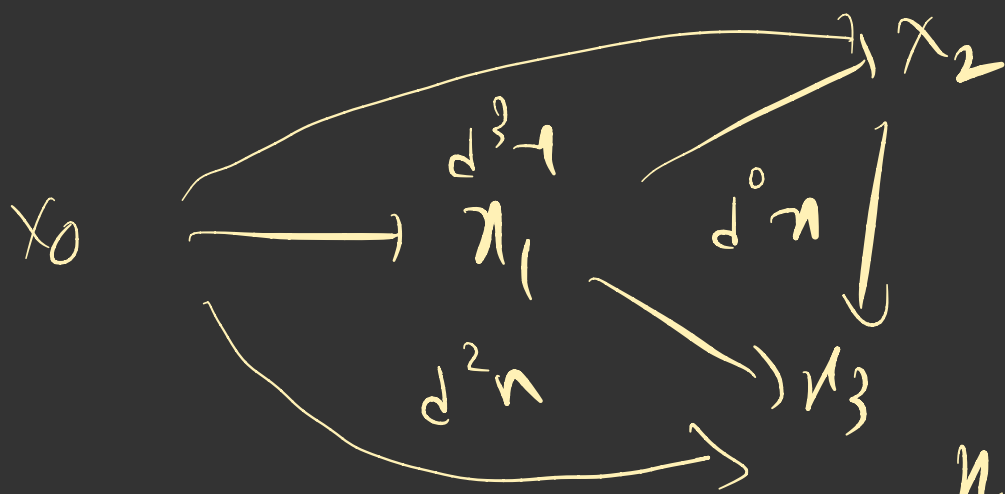
object \rightarrow path \rightarrow

morphisms \rightarrow paths in X

2-morph \rightarrow homotopies between

higher morph \rightarrow higher homot

$$X: SK_1(\Delta^3) \rightarrow X$$



$$d^i_n: \partial \Delta^i \rightarrow X$$

corres. to
restriction of $\tilde{\mu}$

to subarray of $SK_1(\Delta^k)$

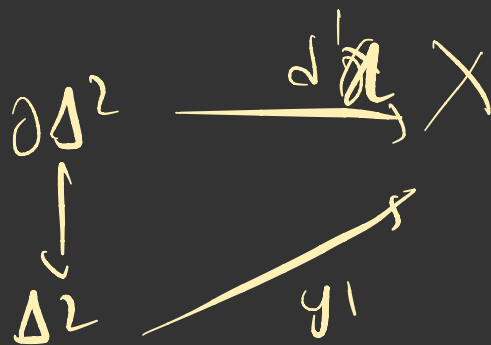
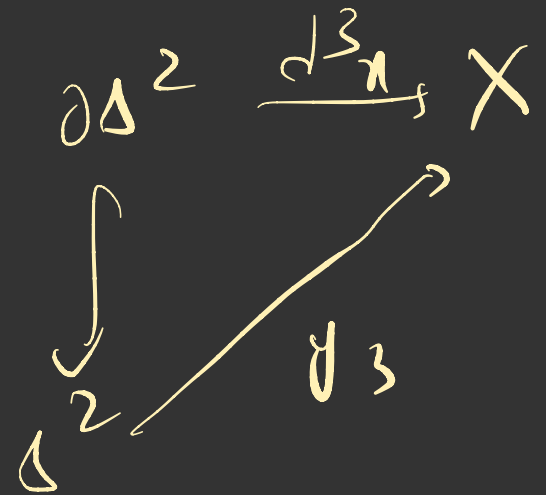
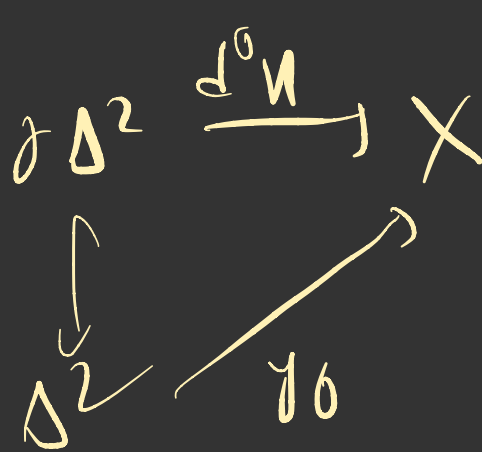
$$n = \{0, 1, 2, 3\} \rightarrow \{n\}$$

Joyal's universal
lemma

If $d^0 u, d^3 u$ commut

$d^1 u$ commutes $\Leftrightarrow d^2 u$ comm

Sketch



$d^2 u$ comm
with $\gamma_2 =$

We get a morphism $(y_0, y_1, y_2): \Delta^2 \rightarrow X$

$$y: \Delta^2 \rightarrow X$$

$$y_i = y \cdot d_i$$

$$\begin{array}{ccc} \partial \Delta^2 & \xrightarrow{d^2} & X \\ \downarrow y_2 & \nearrow & \\ & \partial \Delta^2 & \end{array}$$

HOC

Def Two morphisms $f, g: \Delta^2 \rightarrow C$ in an ∞ -category C are homotopic ($f \simeq g$) if \exists a 2-simplex

$\chi: \Delta^2 \rightarrow C$, with boundary (g, f, h)





$$\lambda: h \rightarrow g$$

Lemma The following relations are equivalent to

$$h \approx_1 g \quad \text{if } g|_X = h$$

$$h \approx_2 g \quad \text{if } h|_X = g$$

$$h \approx_2 g \quad \text{if } h|_Y = g$$

$$h \approx_3 g \quad \text{if } h|_Y = g$$

Prop: \cong is an equivalence on $\text{sets} (X, Y)$ for $X, Y \in \mathcal{C}$
 \downarrow
 (homotopy class of $f: X \rightarrow Y$ is a set)
 denoted by $[f]$.

Def: Let \mathcal{C} is an ∞ -category.
 \exists a 1-category $\text{Hom}(\mathcal{C})$ with same obj. as \mathcal{C} and
 morphisms homotopy classes of morph
 in \mathcal{C}
 (comp. -

$$[g \circ f] := [g \circ f], \quad \text{id}_x := [1_x] = [s_0(m)]$$

The (Boardman
Voyt) \rightarrow a unique map

$$X \xrightarrow{\quad} \underline{N(\text{ho}(X))}$$

which is identity on objects

\downarrow which sends $f: x \rightarrow y$ in X with

(1)

$$\tau(X) \simeq \text{ho}(X)$$

Corollary

∞ -category

X is an ∞ -groupoid if

$$\tau(X) \simeq \text{ho}(X) \text{ is a groupoid.}$$

Def:

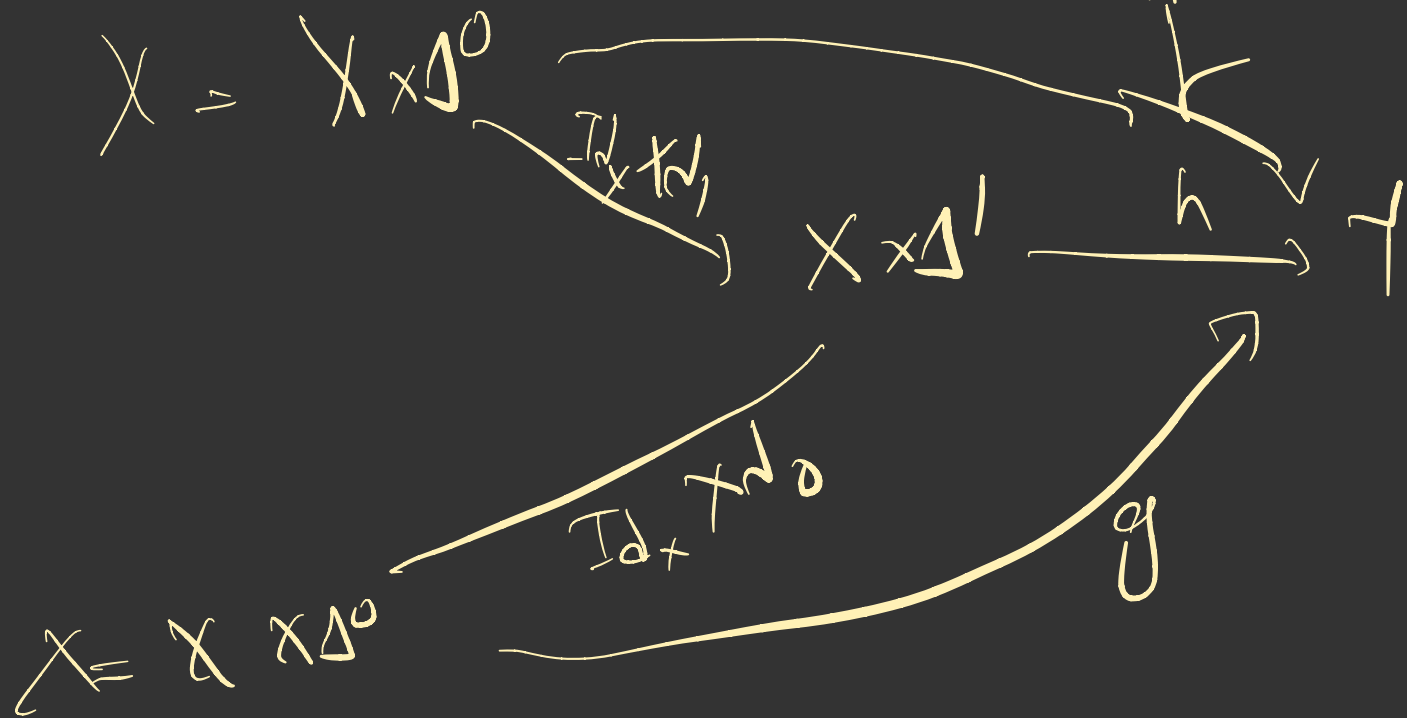
A functor between \mathcal{A} -sets X & Y
is morphic between \mathcal{A} -sets X & Y .

$$h, g : X \rightarrow Y$$

$$h^a : X \times \Delta^1 \rightarrow Y$$

$$h(n, 0) = h(n)$$

$$h(n, 1) = g(n)$$



Ex. For $X \in \text{Top}$ $\mathcal{A} \in \mathcal{B} \text{Cat}$

$$F: \underline{\text{Sing}}(\mathcal{A}) \longrightarrow \underline{N(\mathcal{C})}$$

Def. A morphism $f: x \rightarrow y$ in an ω -category \mathcal{C}
is an equivalence iff $\exists g: y \rightarrow x$ is an
iso in $\text{Ho}(\mathcal{C})$