Nerves, ∞-categories and the Boardman-Vogt construction.

∞-category theory reading group, ntey server

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Plan

If time permits
- Segal category

- The Nerve construction
  - Def and Examples
  - Grothendieck Segal condition
- ∞-categories
  - Def and Examples
  - ∞-groupoids
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- Boardman-Vogt-construction
  - The realization functor
  - Joyal's coherence lemma and the Homotopy cat. const.

Def: The nerve associated to a category is the s.set.

$$NC_n := \mathrm{Hom}_{\mathrm{Cat}}([n], C)$$

Cat → category of small categories

We can think as follows

$NC_n := \{$ strings of n-composable arrows in $C\}$

The degeneracy map $s_i : NC_n \longrightarrow NC_{n+1}$ takes a string of n-composable arrows to a $n+1$-string by attaching an id-map at the ith place.

$$C_0 \xrightarrow{b_1} C_1 \xrightarrow{b_2} \cdots \xrightarrow{b_i} C_i \xrightarrow{b_{i+1}} \cdots \xrightarrow{b_n} C_n$$

$$\Big\downarrow s_i$$

$$C_0 \xrightarrow{b_1} C_1 \xrightarrow{b_2} \cdots \xrightarrow{b_i} C_i \xrightarrow{Id} C_i \xrightarrow{b_{i+1}} C_{i+1} \to \cdots \xrightarrow{b_n} C_n$$

and the face map $d_i : NC_n \longrightarrow NC_{n-1}$ composes the ith and the $(i+1)$th arrows for $0<i<n$

and if $i=0,n$, delete ith object.

$$C_0 \xrightarrow{h_1} C_1 \xrightarrow{h_2} C_2 \to \cdots \to C_{i-1} \xrightarrow{h_i} C_i \xrightarrow{h_{i+1}} C_{i+1} \to \cdots \xrightarrow{h_n} C_n$$

$$\Big\downarrow d_i \ (0 < i < n)$$

$$C_0 \xrightarrow{h_1} C_1 \xrightarrow{h_2} C_2 \to \cdots \to C_{i-1} \xrightarrow{h_{i+1} \circ h_i} C_{i+1} \xrightarrow{h_{i+2}} \cdots \xrightarrow{h_n} C_n$$

We've a fully faithful functor

$$i : \Delta \longrightarrow Set.$$

The nerve functor is just evaluation along i

$$N = i^* : Cat \longrightarrow sSet$$

By Kan's theorem, it has a left adjoint!

$\tau : sSet \longrightarrow Cat$

$(\tau, N) : sSet \rightleftarrows Cat$

Def: For a s.set X, the object are 0-simplices (i.e. $\Delta^0 \to X$) and morphisms are 1-simplices (i.e. $\Delta^1 \longrightarrow X$)

$A \xrightarrow{f} B$ (an edge in X)

the faces of f are given by target $d_0 f = B$ & source $d_1 f = A$

$$
\begin{tikzcd}
\Delta^0 \arrow[r, "d_0"] \arrow[rr, bend left, "A"] & \Delta^1 \arrow[r, "F"] & X \\
& \Delta^0 \arrow[u, "d^1"] \arrow[ur, "B"'] &
\end{tikzcd}
$$

For an object of X, the degenerate edge $s_0(x)$ is the morphism $x \to x$

Recall: For $n \in \mathbb{Z}^{\geq 0}$ Δ^n is the s-set which is given by the following construction

$$\left([k] \in \Delta\right) \longmapsto \left(\mathrm{Hom}_{\Delta}([k],[n])\right)$$

Remark For $n \in \mathbb{Z}^{\geq 0}$ $N([n])$ can be identified with

Δ^n.

For a fin. totally ordered set E we denote

$$\Delta^E = N(E)$$

Now $\partial \Delta^n = \bigcup_{E \subsetneq [n]} \Delta^E \subset \Delta^n$

$$\Lambda^n_k = \bigcup_{k \in E \subsetneq [n]} \Delta^E \subset \Delta^n \qquad n \geq 1, \; 0 \leq k \leq n$$

Recall:

$\mathrm{Spine}[n]$ is a s.set of Δ^n whose k-simplices are monotone maps $\sigma : [k] \longrightarrow [n]$ with the $\sigma(k) \leq \sigma(0) + 1$

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

$$\mathrm{Spine}[n] = \bigcup_{0 \leq j \leq n} \Delta^{\{j, j+1\}} \subset \Delta^n$$

Def: A simp. obj. X in a category $\mathcal{C}$ is a s.set internal to $\mathcal{C}$.

E.g. (Čech Nerve)

Let's consider a category $\mathcal{C}$ with pullbacks and $U \to Z \in \mathrm{Mor}(\mathcal{C})$.

The Čech nerve is a simp. obj. in $\mathcal{C}$

$$C(U) := \left(\cdots U \times_Z U \times_Z U \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix} U \times_Z U \rightrightarrows U \right)$$

Def: A triangle in a sSet X is a map

$$f : \partial\Delta^2 \longrightarrow X$$

It can be visualized (high) with

$f, g, h \in \mathrm{Mor}(X)$, with

target of f coinciding with the source of g, source of f and h are the same,

g and h have the same target

$$\partial\Delta^2 := \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,2\}}$$

f corresponds to the map $\Delta^1 \simeq \Delta^{\{0,1\}} \subset \partial\Delta^2 \to X$

g " " $\Delta^1 \simeq \Delta^{\{1,2\}} \subset \partial\Delta^2 \to X$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

h ———— $\partial \quad \Lambda^2_1 = \mathrm{Spine}(2) \to X$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd}$$

Def: A triangle (f, g, h) is said to commute if $\exists$ a morphism $c : \Delta^2 \to X$ whose restricted to boundary coincides with (f, g, h)

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "{(f,g,h)}"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, "c"'] & \end{tikzcd}$$

Grothendieck - Segal condition

A s.set X satisfies the Grothendieck-Segal condition if the restrictions along the inclusion $\mathrm{Spine}(n) \subset \Delta^n$ induces a bijection

$$\mathrm{Hom}(\Delta^n, X) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Sp}^n, X)$$

$\forall n \in \mathbb{Z}^{\geq 2}$

Consequence If $X \simeq NC$

$$\mathrm{Hom}(\Delta^n, N(C)) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Sp}^n, N(C))$$

Prop. The nerve functor is fully faithful.

Proof.
$$G : \mathrm{Hom}_{\mathrm{Cat}}(C, D) \to \mathrm{Hom}_{\mathrm{Sset}}(NC, ND)$$

Alt. Sketch This also follows from the

fact that the nerve construction gives a 2-coskeletal s.set.

Brief term

If on the simplex set Δ we have a subset. $\Delta_{\leq n} \subset \Delta_n$

$\to$ objects are $[0], \dots, [n]$

Then $\Delta|_{\leq n} \hookrightarrow \Delta$ induces a truncation functor

$$tr_n : sSet \longrightarrow sSet_{\leq n} = [\Delta^{op}_{\leq n}, Set]$$

It has fully faithful left adj

$$sk_n : sSet_{\leq n} \longrightarrow sSet$$

& fully faithful right adj:

$$cosk_n : sSet_{\leq n} \longrightarrow sSet$$

Def: S-sets isomorphic to objects in the range of $cosk_n$ are n-skeletal

$$Hom_{sSet}(NC, ND) \cong Hom_{sSet_{\leq 2}}\left(NC|_{\Delta^{op}_{\leq 2}}, ND|_{\Delta^{op}_{\leq 2}}\right)$$

Prop: The following conditions are equivalent: $X \in \mathrm{sSet}$

(i) $\exists$ a small cat C and an iso-

$$X \simeq N(C)$$

(ii) $X \longrightarrow N(\tau(X))$ is invertible

(iii) X satisfies the Grothendieck-Segal-condition.

Defining ∞-cat

Def: A sset is a Kan-complex if every horn $\Lambda^n_k \longrightarrow X$ $(0 \le k \le n)$ can extended to the n-simplex.

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

Fact: The category of Kan complexes is Cartesian closed.

(fin. prod.
$a \in \mathrm{Obj}(\mathcal{C})$
$- \times a : C \longrightarrow C$ has a
right adj)

Def: An ∞-category is a weak Kan-complex i.e. an extension of $\Lambda^n_k \to X$ for $0<k<n$

to $\Delta^n \to X$ exist

Equivalently: Inner Λ^n_k $(0<k<n)$ have fillers
horn

Further Equivalent def: An ∞-cat is a sset with right lifting property for inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ $0<k<n$

Consequence: $\mathrm{Fun}(\Delta^n, X) \longrightarrow \mathrm{Fun}(\Lambda^n_i, X)$

is surjective if X is an ∞-cat

- For an ∞-category C, objects are vertices $x \in C_0$, morphisms are 1-simplices in C_1.

If $f \in C_1$, $f : X \to Y$, then $X = d_1(f)$ and $Y = d_0(f)$.

- A degen. edge for $X \in C_0$, $s_0(X)$ identity morphism for X.

Def: (∞-groupoids)

An ∞-groupoid is an ∞-category C where every morphism is invertible

$$\forall\ f: X \to Y \in \mathrm{Mor}(C)$$

$$\exists\ g: Y \to X,\ h: Y \to X \text{ s.t.}$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, equal] & & X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, "h"] \arrow[rr, equal] & & Y \end{tikzcd}$$

Examples of ∞-categories

(1) For a small category C, NC is an ∞-category.

Obj $\to$ Objects of C
Mor $\to$ Mor of C

(2) For $X \in Top$, $Sing(X)$ is an ∞-cat.

Identify $|\Delta^n|$ with the hypercube $[0,1]^n$

Obj $\to$ points of X
Morph $\to$ cont. paths $f: [0,1] \to X$
source $f(0)$
target $f(1)$

$$\mathrm{Id}^{x \in X}\colon [0,1] \longrightarrow X \quad \text{taking value at } x$$

<u>Fact</u> We've the following adjunction formula.

$$\mathrm{Hom}(|K|, X) \simeq \mathrm{Hom}(K, \mathrm{Sing}(X))$$

Surjection $\Lambda^n_k \subset \Delta^n$ along

$$\mathrm{Hom}(\Delta^n, \mathrm{Sing}(X)) \longrightarrow \mathrm{Hom}(\Lambda^n_k, \mathrm{Sing}\, X)$$

Prop: Every Kan complex is an ∞-groupoid

Proof: $f: A \to B \in \mathrm{Mor}(X)$ $\quad X \in \underline{\text{Kan}}$

$\exists$ a unique morphism

$$\Lambda^2_0 \longrightarrow A$$

Sending non-degen. 1 simplex $\Delta^{\{0,1\}}$ to f

" " $\qquad$ " " $\qquad \Delta^{\{0,2\}}$ to 1_X

$$\begin{tikzcd} & y \arrow[dr, "?"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & & x \end{tikzcd}$$

Let Grpd be the category of groupoids. Then we have the following comm. diagrams

$$\begin{tikzcd} \mathrm{Grpd} \arrow[r] \arrow[d, "N"'] & \mathrm{Cat} \arrow[d, "N"] \\ \mathrm{Kan} \arrow[r] & \mathrm{sSet} \end{tikzcd}$$

Construct: Let's define a set $X := \Delta^{op} \to \mathrm{Set}$

$$\Delta^{op} \xrightarrow{op} \Delta^{op} \xrightarrow{X} \mathrm{Set}$$

For $\alpha : [m] \to [n]$ in Δ, the morphism $op(\alpha) : [m] \to [n]$ given by

$$op(\alpha)(j) = n - \alpha(m-j)$$

Prop: For a small category C,

$$N(C^{op}) = N(C)^{op}$$

n-simplices of NC are

diagrams

$$c_0 \xrightarrow{h_1} c_1 \to \cdots \xrightarrow{h_n} c_n \quad \text{in } C$$

Then diagram gives a construction of

$$c_n \xrightarrow{h_n} c_{n-1} \to \cdots \xrightarrow{h_1} c_0 \quad \text{in } C^{op}$$

$$N(C^{op}) = N(C)^{op} \quad \text{up to an iso}$$

Boardman–Vogt construction

Def: For any $X \in \mathrm{Top}$ define $\pi_{\leq 1}(X)$ to be the fundamental groupoid of X with

Obj → points in X

For $x, y \in X$, Hom$_{\pi_{\leq 1}(X)}(x, y)$ can be identified with homotopy classes of cont. paths $p : [0,1] \to X$ with $p(0) = x$, $p(1) = y$

– composition is by concatenation of paths

Refinement

objects → points in X

morphisms → paths in X

2-morphisms → homotopies between paths

higher morphisms → higher homotopies

$x : Sk_1(\Delta^3) \longrightarrow X$

$$\begin{tikzcd}
 & & x_2 \arrow[d] \\
x_0 \arrow[urr, bend left] \arrow[r] \arrow[rr, bend right] & x_1 \arrow[ur] \arrow[r] & x_3
\end{tikzcd}$$

$d^3 x$, $d^0 x$, $d^2 x$

$d^i x : \partial\Delta^2 \to X$ corres.. to restriction of x to subcomplex of $Sk_1(\Delta^3)$

$n = \{0,1,2,3\} - \{n\}$

Joyal's extension theorem

If $\underbrace{d^0 x,\ d^3 x}_{\text{then}}$ commute

$d^1 x$ commutes $\Leftrightarrow$ $d^2 x$ commutes

Sketch

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "d^0 x"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, "y_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta^2 \arrow[r, "d^3 x"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, "y_3"'] & \end{tikzcd}$$

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "d^1 x"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, "y_1"'] & \end{tikzcd}$$

$d^2 x$ commutes with $y_2 =$

We get a morphism $(y_0, y_1, y_3): \Lambda^3_2 \to X$

$y: \Delta^3 \to X$. $y_2 = y \cdot d_2$

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "d^2 x"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, "y_2"'] & \end{tikzcd}$$

HO(C)

Def Two morphisms $f, g: x \to y$ in an ∞-category C are homotopic ($f \simeq g$) if $\exists$ a 2-simplex $\sigma: \Delta^2 \to C$, with boundary (g, f, id_x)

$$\begin{tikzcd} & {} \arrow[d, "g"] \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

$\lambda : f \to g$

Lemma The following relations are equivalent to

$f \simeq g$ iff $g|_X = f$

$f \simeq_1 g$ iff $f|_X = g$

$f \simeq_2 g$ iff $1_Y f = g$

$f \simeq_3 g$ iff $1_Y g = f$

Exer: $\simeq$ is an equivalent relation on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for $X, Y \in \mathcal{C}$ (∞-category)

Homotopy class of $f: X \to Y$ is denoted by $[f]$.

Def: Let $\mathcal{C}$ is an ∞-category.

$\exists$ a 1-category $Ho(\mathcal{C})$ with same objects as $\mathcal{C}$ and morphisms homotopy classes of morphisms in $\mathcal{C}$.

Composition

$$[g] \circ [f] := [g \circ f], \quad id_X := [id_X] = [s_0(x)]$$

Thm (Boardman Vogt) $\exists$ a unique map

$$X \longrightarrow N(ho(X))$$

which is identity on objects & which sends $f: x \to y$ in X to its $[f]$

$$\tau(X) \simeq Ho(X)$$

Corollary ∞-category X is an ∞-groupoid iff $\tau(X) \simeq Ho(X)$ is a groupoid.

Def: A functor between ∞-category X & Y is morphism between ∞ s.sets $X \to Y$.

$$f, g : X \longrightarrow Y$$

$$h : X \times \Delta^1 \longrightarrow Y \qquad h(x,0) = f(x)$$

$$h(x,1) = g(x)$$

$$\begin{tikzcd}
X = X \times \Delta^0 \arrow[dr, "Id_X \times d_1"] \arrow[drr, bend left] & & \\
& X \times \Delta^1 \arrow[r, "h"] & Y \\
X = X \times \Delta^0 \arrow[ur, "Id_X \times d_0"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

E.g. For $X \in \mathrm{Top}$ and $C \in \mathrm{Cat}$

$$F : \underline{\mathrm{Sing}}(X) \longrightarrow \underline{N(C)}$$

Def. A morphism $f : x \to y$ in an ∞-category C is an equivalence if $[f] : x \to y$ is an iso in $\mathrm{Ho}(C)$